Competency-Based Questions — Application of Derivatives

(Class XII Mathematics, Chapter 6)

Competency-based questions test conceptual understanding, application to real/unfamiliar situations, and reasoning — rather than plain formula recall. They are grouped by sub-topic following the NCERT syllabus flow: Rate of Change → Increasing/Decreasing Functions → Maxima and Minima → Absolute Maximum/Minimum on a Closed Interval.

Section A: Rate of Change of Quantities

Q1. (Real-life application — inflating balloon) A weather balloon, which always remains spherical, is being inflated at a rate of 900 cm^3 of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm. Explain why the *rate of increase of radius* slows down even though gas is pumped in at a constant rate.

Answer: $V = (4/3)\pi r^3$ $dV/dt = 4\pi r^2 (dr/dt)$

Given $dV/dt = 900$ cm^3/s, $r = 15$ cm: $900 = 4\pi(15)^2(dr/dt) = 900\pi\,(dr/dt)$ $dr/dt =$ **$1/\pi$ cm/s**

Interpretation: As the balloon gets bigger, the same 900 cm^3 of gas has to spread over a much larger surface area ($4\pi r^2$ grows as the square of r), so each additional cm of radius "costs" more volume. This is why $dr/dt = (dV/dt)/(4\pi r^2)$ decreases as r increases, even though dV/dt is constant — a good illustration of how related rates aren't simply proportional.

Q2. (Application — economics) The total cost of producing x units of a product is $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$ (in ₹). A factory manager wants to know: is it cheaper (per additional unit) to increase production from 2 to 3 units, or from 50 to 51 units? Use the marginal cost concept to answer, and explain what "marginal cost" tells the manager that average cost does not.

Answer: $MC(x) = dC/dx = 0.015x^2 - 0.04x + 30$

$MC(3) = 0.015(9) - 0.04(3) + 30 = 0.135 - 0.12 + 30 = 30.015$ $MC(51) = 0.015(2601) - 0.04(51) + 30 = 39.015 - 2.04 + 30 = 66.975$

Since $MC(51) > MC(3)$, it is **more expensive** (per additional unit) to increase production near the 50-unit mark than near the 2-unit mark. *Interpretation:* Marginal cost tells the manager the *instantaneous* cost of producing one more unit at a given production level — critical for deciding whether to scale up production further — whereas average cost only gives a cost per unit averaged over *all* units already produced, hiding how costs behave at the margin.

Q3. (Application — geometry in motion, ladder problem) A 5 m ladder leans against a wall. The foot of the ladder is pulled away from the wall at 2 cm/s. Find how fast the top of the ladder is sliding down the wall when the foot is 4 m from the wall. Why does the top slide down *faster and faster* as the foot moves further from the wall (even though the foot moves at constant speed)?

Answer: Let x = distance of foot from wall, y = height of top on wall. $x^2 + y^2 = 25$. $2x(dx/dt) + 2y(dy/dt) = 0 \Rightarrow dy/dt = -(x/y)(dx/dt)$

At $x = 4$: $y = \sqrt{(25-16)} = 3$ $dy/dt = -(4/3)(2) =$ **$-8/3$ cm/s** (negative sign shows y is decreasing)

Interpretation: As x increases, y decreases, so the ratio x/y grows rapidly — meaning the same horizontal speed dx/dt translates into a larger vertical speed near the end (when the ladder is almost flat on the ground, $y \to 0$ and $dy/dt \to -\infty$ in principle). This is a classic case where a constant input rate produces an accelerating output rate due to the changing geometry.

Section B: Increasing and Decreasing Functions

Q4. (Application — public health/epidemic modelling) The number of people infected during an outbreak is modelled (in suitable units) by $f(t) = t^3 - 3t^2 + 4t$ for $t \geq 0$ (t in weeks). Show that infections are *always* rising, never falling, in this model. What real-world limitation does this reveal about the model?

Answer: $f'(t) = 3t^2 - 6t + 4 = 3(t^2 - 2t + 1) + 1 = 3(t - 1)^2 + 1 > 0$ for all t

Since $f'(t) > 0$ always, f is **strictly increasing on the entire domain** — infections never decrease under this model. *Real-world limitation:* Real epidemics eventually peak and decline (as susceptible people run out, or interventions take effect) — a cubic model with always-positive derivative cannot capture this turning point. This shows why polynomial models are often only used for a *limited window* of real data, not indefinitely.

Q5. (Reasoning/interpretation — business cycles) A company's profit rate (in ₹ lakh) is modelled by $f(x) = \sin x + \cos x$ for $x \in [0, 2\pi]$ (x measured in suitable time units). Identify the intervals where profit is increasing and where it's decreasing, and interpret what happens to the business at $x = \pi/4$ and $x = 5\pi/4$.

Answer: $f'(x) = \cos x - \sin x = 0$ gives $x = \pi/4, 5\pi/4$

- f is increasing on $[0, \pi/4)$ and $(5\pi/4, 2\pi]$ ($f'(x) > 0$)
- f is decreasing on $(\pi/4, 5\pi/4)$ ($f'(x) < 0$)

Interpretation: At $x = \pi/4$, the profit function reaches a **local peak** — business performance is at a temporary high, after which it starts declining. At $x = 5\pi/4$, profit hits a **local trough** and starts recovering. These turning points would correspond to real strategic decision moments — e.g., a company might want to introduce new interventions right around $x = \pi/4$ to prevent the coming decline.

Q6. (Common misconception check) A student claims: "$f'(x) \geq 0$ for all x in an interval is required for f to be called increasing there — if $f'(x) = 0$ anywhere, the function cannot be increasing." Use $f(x) = x^3$ on $\mathbb{R}$ to correct this misconception.

Answer: The student's claim is **incorrect**. For $f(x) = x^3$, $f'(x) = 3x^2$, which equals 0 exactly at $x = 0$ but is positive everywhere else. By the theorem (f increasing on $[a,b]$ if $f'(x) \geq 0$ for each x in (a,b)), x^3 is still (strictly) increasing on all of $\mathbb{R}$ — a single isolated point where the derivative is zero (a "momentary pause" in the rate of increase) does not break the overall increasing trend, because the function value still consistently rises as x increases through that point. This clarifies that $f'(x) \geq 0$ (not strictly >0) is the correct condition for increasing, and equality at isolated points is allowed.

Section C: Maxima and Minima — Conceptual

Q7. (Conceptual/graph-based) A student says: "A function must be differentiable at a point for that point to be a local maximum or minimum." Use $f(x) = 3 + |x|$ to show this is false, and explain how the *first derivative test* still applies even when the second derivative test cannot be used.

Answer: The claim is **false**. For $f(x) = 3 + |x|$, f is not differentiable at $x = 0$ (sharp corner), yet $x = 0$ is clearly a point of local minima ($f(0) = 3$ is the least value nearby).

Since $f''(0)$ doesn't exist, the **second derivative test fails**, but the **first derivative test** still works: to the left of 0, $f'(x) = -1 < 0$ (decreasing); to the right of 0, $f'(x) = 1 > 0$ (increasing). Since f' changes sign from negative to positive at $x = 0$, it is a point of local minima — even without differentiability at that exact point. This shows the first derivative test is more broadly applicable, since it only requires *continuity* at the critical point, not differentiability there.

Q8. (Application — military/defense scenario, distance minimization) An enemy helicopter flies along the path $y = x^2 + 7$. A soldier positioned at $(3, 7)$ wants to know the closest the helicopter ever gets to him. Find this minimum distance, and explain why minimizing the *square* of the distance (rather than the distance itself) is a valid and often easier strategy.

Answer: $\text{Distance}^2 = (x-3)^2 + (x^2+7-7)^2 = (x-3)^2 + x^4$ Let $f(x) = (x-3)^2 + x^4$ $f'(x) = 2(x-3) + 4x^3 = 2(x-1)(2x^2 + 2x + 3)$

$f'(x) = 0$ gives $x = 1$ (the quadratic factor has no real roots). $f(1) = (1-3)^2 + 1^4 = 4 + 1 = 5$ Minimum distance $= \sqrt{5}$

Why square the distance: Since distance is always non-negative, minimizing distance^2 gives the same critical points as minimizing distance directly (the square-root function is increasing, so it preserves the location of the minimum), but avoids dealing with a messy square root while differentiating — a common and valid optimization shortcut.

Q9. (Application — packaging/manufacturing) An open-topped box is made from a 3 m × 8 m rectangular aluminium sheet by cutting equal squares (side x) from each corner and folding up the sides. Find the size of the square that maximizes the box's volume, and explain why x cannot be too large or too small in this real context (i.e., why the domain of x matters physically).

Answer: $V(x) = x(3-2x)(8-2x) = 4x^3 - 22x^2 + 24x$ $V'(x) = 12x^2 - 44x + 24 = 4(x-3)(3x-2)$ $V'(x) = 0$ gives $x = 3$ or $x = 2/3$. Since $x = 3$ makes the width $(3-2x)$ negative (impossible), reject it.

At $x = 2/3$: $V''(x) = 24x - 44$, $V''(2/3) = 16 - 44 = -28 < 0 \rightarrow$ local maximum. Maximum volume = $V(2/3) = 200/27\ m^3$

Physical domain reasoning: x must satisfy $0 < x < 1.5$ (since the shorter side is 3 m, cutting a square of side ≥ 1.5 m would leave no width at all). This shows that in real optimization problems, the mathematically "valid" critical points must always be checked against the physically meaningful domain — a purely mathematical answer ($x = 3$) can be nonsensical in context.

Q10. (HOTS — engineering, structural design) A cone-shaped water tank (vertex down) has semi-vertical angle $\tan^{-1}(0.5)$. Water flows in at $5\ m^3$/hour. Find how fast the water level is rising when the depth is 4 m. Why does the rate at which the level rises change over time even though the inflow rate is constant?

Answer: $r = h/2$ (from $\tan \alpha = r/h = 0.5$), $V = (1/3)\pi r^2 h = \pi h^3/12$ $dV/dt = (\pi h^2/4)(dh/dt)$

At $h = 4$, $dV/dt = 5$: $5 = (\pi \cdot 16/4)(dh/dt) = 4\pi(dh/dt)$ $dh/dt = 5/(4\pi) =$ **35/88 m/h** (using $\pi = 22/7$)

Interpretation: Because the cone is narrower near the bottom and wider near the top, the same volume of water raises the level *faster* when the tank is nearly empty (small h, small cross-section) and *slower* as it fills up (large h, large cross-section) — this is why $dh/dt = (dV/dt)/(\pi h^2/4)$ depends inversely on h^2, even with constant inflow.

Q11. (Application — physics of shadows) A 2 m tall man walks away from a 6 m lamp post at 5 km/h. Find the rate at which the length of his shadow is increasing. Explain intuitively (using similar triangles) why the shadow's growth rate is a fixed *fraction* of the man's own walking speed, rather than equal to it.

Answer: Using similar triangles $\Delta MSN \sim \Delta ASB$: $MS/AS = MN/AB$ gives $AS = 3s$ (where s = shadow length), and since $AM = AS - MS = 2s = l$ (distance walked): $l = 2s \Rightarrow dl/dt = 2(ds/dt)$ Since $dl/dt = 5$ km/h: $ds/dt =$ **5/2 km/h**

Interpretation: The similar-triangle ratio (height of man : height of lamp = 2:6 = 1:3) fixes the relationship between how far the man walks and how much his shadow grows — since $AM = 2s$ here, the shadow grows at exactly half the man's walking speed. This shows how a *fixed geometric ratio* (not the absolute distance) governs the relationship between two related rates.

Section D: Absolute Maximum and Minimum on Closed Intervals

Q12. (Application — resource optimization) Two positive numbers have a sum of 24. What should the numbers be so that their product is as large as possible? Justify why the strategy of "split evenly" works using the second derivative test, and explain the broader principle this illustrates (relevant to resource allocation).

Answer: Let the numbers be x and 24 − x. $P(x) = x(24-x) = 24x - x^2$ $P'(x) = 24 - 2x = 0 \Rightarrow x = 12$ $P''(x) = -2 < 0 \rightarrow$ local maximum at x = 12

So the numbers are **12 and 12**, giving maximum product 144.

Broader principle: When a *fixed total* is split between two variables whose product (or a similar symmetric combined quantity) is to be maximized, the optimal split is often the *equal* one — a principle used broadly in resource allocation (e.g., splitting a fixed budget or fixed area between two competing uses to maximize a combined benefit).

Q13. (Application — sustainable design) Of all closed cylindrical cans with a fixed volume of 100 cubic units, find the dimensions of the can that uses the *least material* (minimum surface area). Why would a company producing millions of cans care about solving this exact optimization problem?

Answer: Let radius = r, height = h. Volume: $\pi r^2 h = 100 \Rightarrow h = 100/(\pi r^2)$ Surface area: $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 200/r$ $dS/dr = 4\pi r - 200/r^2 = 0 \Rightarrow r^3 = 200/(4\pi) = 50/\pi \Rightarrow r = (50/\pi)$^(1/3)

Second derivative: $d^2S/dr^2 = 4\pi + 400/r^3 > 0$ always → this critical point is a minimum.

Real-world relevance: For a company manufacturing millions of cans, even a tiny reduction in material per can (achieved by choosing the mathematically optimal r and h) translates into massive savings in raw material cost and, importantly, reduced material waste — directly connecting a calculus optimization result to cost efficiency and environmental sustainability.

Q14. (Application — comparing intervals, closed-interval theorem) Explain, using the function $f(x) = x^3 - 15x^2 + 36x + 1$ on the interval [1, 5], why checking *only* the critical points (where $f'(x) = 0$) is not enough to find the absolute maximum and minimum — you must also check the endpoints. Illustrate with the actual computation.

Answer: $f'(x) = 6x^2 - 30x + 36$ wait — actually using given function $2x^3-15x^2+36x+1$: $f'(x) = 6x^2-30x+36 = 6(x-2)(x-3)$ Critical points: x = 2, x = 3 (both inside [1,5])

f(1) = 24, f(2) = 29, f(3) = 28, f(5) = 56

The **absolute maximum (56)** occurs at x = 5 — an **endpoint**, not a critical point! The absolute minimum (24) occurs at x = 1 — also an endpoint.

Explanation: Critical points only locate *local* turning points within the open interval; they say nothing about how the function behaves at the boundaries of a closed interval. Since a continuous function on a closed interval always attains its absolute max/min (by Theorem 5), and these could occur either at critical points or endpoints, both must be checked and compared — skipping the endpoints here would have led to a wrong conclusion that 29 (a local maximum) is the absolute maximum, when 56 (at the endpoint) is actually far larger.

Section E: Integrated / Case-Study Style Question

Q15. (Case Study — combined rate of change and optimization) A car starts from point P at $t = 0$ and stops at point Q. Its distance covered is $x(t) = t^2(2 - t/3)$ metres.

(a) Find the velocity function $v(t)$, and determine when the car is at rest (i.e., at P and at Q). (b) Find the total distance between P and Q. (c) Was the car's velocity always increasing, always decreasing, or neither, throughout the journey? What does this tell you about the car's motion (accelerating vs. decelerating phases)?

Answer: (a) $v(t) = dx/dt = 4t - t^2 = t(4-t)$. $v = 0$ at $t = 0$ (point P) and $t = 4$ (point Q).

(b) $x(4) = 4^2(2 - 4/3) = 16(2/3) =$ **32/3 m**. The car reaches Q after 4 seconds, having covered 32/3 m.

(c) $v(t) = 4t - t^2$, so $v'(t) = 4 - 2t = 0$ at $t = 2$. Since $v'(t) > 0$ for $t < 2$ and $v'(t) < 0$ for $t > 2$, velocity **increases from $t = 0$ to $t = 2$** (car accelerating) and **decreases from $t = 2$ to $t = 4$** (car decelerating), reaching zero again at Q.

Interpretation: This is a complete physical picture of the car's trip — it starts at rest, speeds up for the first half of the journey (0 to 2 seconds), reaches a maximum speed at $t = 2$ seconds, then decelerates smoothly to come to rest exactly at Q. This combines rate-of-change (velocity as a derivative) with increasing/decreasing analysis (analyzing $v'(t)$, i.e., acceleration) in a single real motion problem.

Quick Reference: Key Ideas Behind These Questions

Concept	Competency Being Tested
Rate of change	Relating simultaneous change in geometric/physical quantities via chain rule
Increasing/decreasing functions	Recognizing trends, turning points, and their real-world meaning
Critical points & first derivative test	Locating maxima/minima even without differentiability
Second derivative test	Efficiently confirming nature of a critical point
Optimization (maxima/minima)	Translating real constraints (cost, material, distance, resources) into calculus problems
Absolute max/min on closed intervals	Understanding why endpoints matter, not just critical points
Combined rate + optimization problems	Building a full physical/motion picture using multiple derivative concepts together

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This set can be used for classroom discussion, assignments, or as a model for framing your own competency-based questions on this chapter.

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