RELATIONS AND FUNCTIONS

Complete Study Material

Class XII Mathematics — Chapter 1

Notes • MCQs • Short Answer • Long Answer • Homework — all with explained solutions

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Contents

1. Concept Notes

1.1 Relations

A relation R from a set A to a set B is any subset of A × B. If (a, b) ∈ R, we write a R b, meaning 'a is related to b'.

- Empty relation: R = φ ⊂ A × A (no element related to any element).
- Universal relation: R = A × A (every element related to every element).
- Both empty and universal relations are called trivial relations.

1.2 Types of Relations on a Set A

Type	**Condition**	**Quick Example**
Reflexive	(a, a) ∈ R for every a ∈ A	a = a
Symmetric	$(a_1,a_2) \in R \Rightarrow (a_2,a_1) \in R$	$L_1 \perp L_2 \Rightarrow L_2 \perp L_1$
Transitive	$(a_1,a_2) \in R$ and $(a_2,a_3) \in R \Rightarrow (a_1,a_3) \in R$	$a \leq b, b \leq c \Rightarrow a \leq c$
Equivalence	Reflexive + Symmetric + Transitive, all together	Congruence of triangles

Tip: To disprove reflexive/symmetric/transitive, one counter-example from the set is enough. To prove, a general argument valid for all elements is required.

1.3 Equivalence Classes
If R is an equivalence relation on X, then R partitions X into disjoint subsets called equivalence classes. The equivalence class of a, denoted [a], is the set of all elements related to a.

- Any two equivalence classes are either identical or completely disjoint.
- The union of all equivalence classes equals X.
- Example: For R = {(a,b) : 2 divides a−b} on Z, the classes are [0] = all even integers and [1] = all odd integers.

1.4 Types of Functions

Term	Definition
One-one (Injective)	$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, i.e. distinct inputs give distinct outputs.
Onto (Surjective)	Every $y \in Y$ has some $x \in X$ with $f(x) = y$, i.e. Range(f) = Y (codomain).
Bijective	Both one-one and onto.
Invertible	f is invertible ⇔ f is bijective. Then f^{-1} exists with $f\circ f^{-1} = I\_Y$ and $f^{-1}\circ f = I\_X$.

- Key finite-set fact: For a finite set X, f : X → X is one-one ⇔ f is onto (each implies the other). This is FALSE for infinite sets — e.g. f : N → N, f(x) = 2x is one-one but not onto.

1.5 Composition of Functions

If f : A → B and g : B → C, then gof : A → C is defined by (gof)(x) = g(f(x)) for all $x \in A$.

- Composition is generally NOT commutative: gof ≠ fog in general.
- Composition is associative: (hog)of = ho(gof).
- If f and g are both one-one, gof is one-one; if both onto, gof is onto; if both bijective, gof is bijective.

1.6 Quick Formula / Method Reference

- Proving reflexive: check $(a, a) \in R$ for a generic $a \in A$.
- Proving symmetric: assume $(a, b) \in R$, derive $(b, a) \in R$.
- Proving transitive: assume $(a, b) \in R$ and $(b, c) \in R$, derive $(a, c) \in R$.
- Proving one-one: assume $f(x_1) = f(x_2)$, algebraically show $x_1 = x_2$ (or use the graph — a horizontal line test).
- Proving onto: take arbitrary y in codomain, solve f(x) = y for x, and confirm this x lies in the domain.
- Proving invertible: show bijective, then construct g by solving y = f(x) for x.

2. Multiple Choice Questions

Each question has one correct option. Explanations are provided immediately after each question.

1. **Q1. The relation R = {(a, b) : a − b = 10} in the set A = {1, 2, 3, 4} is:**
   (a) Universal relation
   (b) Empty relation
   (c) Reflexive but not symmetric
   (d) An equivalence relation

Answer: (b) Empty relation. No pair in A × A satisfies a − b = 10 since the maximum possible difference is 4 − 1 = 3. So R = φ.

2. **Q2. Let R = {(a, b) : b = a + 1} on the set {1, 2, 3, 4, 5, 6}. R is:**
   (e) Reflexive
   (f) Symmetric

(g) Transitive

(h) Neither reflexive, symmetric nor transitive

Answer: (d). Not reflexive since $(1,1) \notin R$ $(1 \neq 1+1)$. Not symmetric since $(1,2) \in R$ but $(2,1) \notin R$ $(2 \neq 1+1)$. Not transitive since $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$ $(3 \neq 1+1)$.

3. **Q3. The relation R in R defined as R = {(a, b) : a ≤ b} is:**

(i) Symmetric only

(j) Reflexive and transitive but not symmetric

(k) An equivalence relation

(l) Neither reflexive nor transitive

Answer: (b). Reflexive since $a \leq a$ always. Transitive since $a \leq b$ and $b \leq c \Rightarrow a \leq c$. Not symmetric because, e.g., $2 \leq 3$ but $3 \leq 2$ is false.

4. **Q4. Let R be the relation in N given by R = {(a, b) : a = b − 2, b > 6}. Which is correct?**

(m) $(2, 4) \in R$

(n) $(3, 8) \in R$

(o) $(6, 8) \in R$

(p) $(8, 7) \in R$

Answer: (c). Check (6,8): a=6, b=8, need $a = b-2 = 6$ ✓ and $b > 6$ i.e. $8 > 6$ ✓. So $(6,8) \in R$. (2,4) fails $b > 6$; (3,8) fails $a = b-2$ $(8-2=6\neq3)$; (8,7) fails $b > 6$.

5. **Q5. f : R → R defined by f(x) = 3x is:**

(q) One-one onto

(r) Many-one onto

(s) One-one but not onto

(t) Neither one-one nor onto

Answer: (a). One-one: $f(x_1)=f(x_2) \Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2$. Onto: for any $y \in R$, $x = y/3 \in R$ gives $f(x) = y$. So f is bijective, i.e. one-one onto.

6. **Q6. f : R → R defined by f(x) = x^4 is:**

(u) One-one onto

(v) Many-one onto

(w) One-one but not onto

(x) Neither one-one nor onto

Answer: (d). Not one-one: $f(-1) = 1 = f(1)$ but $-1 \neq 1$. Not onto: negative numbers like -1 in the codomain R have no pre-image since $x^4 \geq 0$ always.

7. **Q7. The number of equivalence relations in the set {1, 2, 3} containing (1, 2) and (2, 1) is:**

(y) 1

(z) 2

(aa) 3

(bb) 4

Answer: (b) 2. The smallest such relation is {(1,1),(2,2),(3,3),(1,2),(2,1)}. Adding any further pair like (2,3) forces (via symmetry and transitivity) the addition of (3,2),(1,3),(3,1), producing the universal relation. So only two equivalence relations exist: this smallest one and the universal relation $A \times A$.

8. **Q8. Let A and B be sets. The function f : A × B → B × A defined by f(a, b) = (b, a) is:**

(cc) Only one-one

(dd) Only onto

(ee) Bijective

(ff) Neither one-one nor onto

Answer: (c) Bijective. One-one: $f(a_1,b_1) = f(a_2,b_2) \Rightarrow (b_1,a_1) = (b_2,a_2) \Rightarrow a_1=a_2, b_1=b_2$. Onto: for any $(b, a) \in B \times A$, the pre-image $(a, b) \in A \times B$ maps to it. Hence bijective.

9. **Q9. If $f : X \to Y$ is invertible, then f must be:**
   (gg) Only one-one
   (hh) Only onto
   (ii) One-one and onto (bijective)
   (jj) Reflexive and symmetric

Answer: (c). By definition, f is invertible iff there exists g with gof = I_X and fog = I_Y — this is possible exactly when f is bijective (one-one and onto).

10. **Q10. The relation R = {(1,1),(2,2),(3,3),(1,2),(2,3)} on {1,2,3} is:**
   (kk) Reflexive but neither symmetric nor transitive
   (ll) Symmetric only
   (mm) Transitive only
   (nn) An equivalence relation

Answer: (a). Reflexive since $(1,1),(2,2),(3,3) \in R$. Not symmetric since $(1,2) \in R$ but $(2,1) \notin R$. Not transitive since $(1,2) \in R$, $(2,3) \in R$, but $(1,3) \notin R$.

3. Short Answer Questions

These carry 2–3 marks each. Concise, complete solutions are given.

1. **Q1. Check whether the relation R = {(a, b) : b = a + 1} on the set {1, 2, 3, 4, 5, 6} is reflexive, symmetric or transitive.**

Solution: Not reflexive: (1,1) needs 1 = 1+1, false. Not symmetric: $(1,2) \in R$ (since 2=1+1) but (2,1) needs 1=2+1, false. Not transitive: $(1,2) \in R$ and $(2,3) \in R$, but (1,3) needs 3=1+1, false, so $(1,3) \notin R$. Hence R is neither reflexive, symmetric, nor transitive.

2. **Q2. Show that the relation R in the set {1, 2, 3} given by R = {(1,2),(2,1)} is symmetric but neither reflexive nor transitive.**

Solution: Symmetric: $(1,2) \in R$ and $(2,1) \in R$, so the condition $(a,b)\in R \Rightarrow (b,a)\in R$ holds for the only pairs present. Not reflexive: $(1,1) \notin R$. Not transitive: $(1,2) \in R$ and $(2,1) \in R$, so we'd need $(1,1) \in R$ for transitivity, but $(1,1) \notin R$.

3. **Q3. Prove that $f : R \to R$ given by $f(x) = 2x$ is one-one and onto.**

Solution: One-one: Let $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. So f is one-one. Onto: Let $y \in R$ be arbitrary. Choose $x = y/2 \in R$. Then $f(x) = 2(y/2) = y$. So every y has a pre-image, hence f is onto. Therefore f is bijective.

4. **Q4. Show that $f : N \to N$ given by $f(x) = x^2$ is one-one but not onto.**

Solution: One-one: $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ (since $x_1, x_2 \in N$ are positive, the negative root is rejected). Not onto: take $y = 2 \in N$ (codomain); there is no natural number x with $x^2 = 2$. So f is not onto.

5. **Q5. If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation.**

Solution: Reflexive: since R_1, R_2 are reflexive, $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, so $(a,a) \in R_1 \cap R_2$. Symmetric: $(a,b) \in R_1 \cap R_2 \Rightarrow (a,b) \in R_1$ and $(a,b) \in R_2 \Rightarrow (b,a) \in R_1$ and $(b,a) \in R_2$ (by symmetry of each) $\Rightarrow (b,a) \in R_1 \cap R_2$. Transitive: $(a,b),(b,c) \in R_1 \cap R_2 \Rightarrow$ both lie in R_1 and in $R_2 \Rightarrow (a,c) \in R_1$ and $(a,c) \in R_2$ (by transitivity of each) $\Rightarrow (a,c) \in R_1 \cap R_2$. Hence $R_1 \cap R_2$ is reflexive, symmetric and transitive, i.e. an equivalence relation.

6. **Q6. Find gof and fog if $f(x) = \cos x$ and $g(x) = 3x^2$, for $f, g : R \to R$. Are they equal?**

Solution: $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$. At x = 0: $gof(0) = 3(1)^2 = 3$, while $fog(0) = \cos(0) = 1$. Since $3 \neq 1$, gof ≠ fog — composition is not commutative in general.

7. **Q7. Let $f : N \to Y$, $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x+3, \text{ some } x \in N\}$. Show f is invertible and find f^{-1}.**

Solution: For $y \in Y$, $y = 4x+3 \Rightarrow x = (y-3)/4$. Define $g(y) = (y-3)/4$. Then $gof(x) = g(4x+3) = (4x+3-3)/4 = x$, so gof = I_N. Also $fog(y) = f((y-3)/4) = 4\cdot(y-3)/4 + 3 = y-3+3 = y$, so fog = I_Y. Since both compositions give identity, f is invertible with $f^{-1}(y) = (y-3)/4$.

8. **Q8. Show that the Signum function f : R → R given by f(x) = 1 (x>0), 0 (x=0), −1 (x<0) is neither one-one nor onto.**

Solution: Not one-one: f(2) = 1 = f(5), but 2 ≠ 5 — many different positive values map to the same output 1. Not onto: the codomain is R, but the range of f is only {−1, 0, 1}; a value like 0.5 in R has no pre-image. Hence f is neither one-one nor onto.

4. Long Answer Questions

These carry 5 marks each and require complete step-by-step justification.

1. **Q1. Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides a − b} is an equivalence relation. Also write the equivalence classes.**

Solution: Reflexive: For any a ∈ Z, a − a = 0, and 2 divides 0. So (a,a) ∈ R for all a. Symmetric: Let (a,b) ∈ R, so 2 divides (a−b), meaning a−b = 2k for some integer k. Then b−a = −2k = 2(−k), so 2 divides (b−a), giving (b,a) ∈ R. Transitive: Let (a,b) ∈ R and (b,c) ∈ R. Then a−b = $2k_1$ and b−c = $2k_2$ for integers k_1, k_2. Adding, a−c = (a−b)+(b−c) = $2(k_1+k_2)$, which is divisible by 2. So (a,c) ∈ R. Since R is reflexive, symmetric and transitive, R is an equivalence relation. Equivalence classes: all integers related to 0 form the class [0] = set of even integers {..., −4,−2,0,2,4,...}; all integers related to 1 form [1] = set of odd integers {...,−3,−1,1,3,5,...}. These two classes are disjoint and their union is Z.

2. **Q2. Let A = {1,2,3,...,9} and R be the relation in A × A defined by (a,b) R (c,d) if a + d = b + c for (a,b),(c,d) ∈ A × A. Prove that R is an equivalence relation.**

Solution: Reflexive: (a,b) R (a,b) requires a+b = b+a, which is always true. So R is reflexive. Symmetric: Suppose (a,b) R (c,d), i.e. a+d = b+c. Rearranging, c+b = d+a, i.e. b+c = a+d ⇒ this exactly says c+b = d+a, i.e. (c,d) R (a,b). So R is symmetric. Transitive: Suppose (a,b) R (c,d) and (c,d) R (e,f). Then a+d = b+c ... (i) and c+f = d+e ... (ii). Adding (i) and (ii): a+d+c+f = b+c+d+e. Cancelling c and d from both sides: a+f = b+e, which means (a,b) R (e,f). So R is transitive. Since R is reflexive, symmetric and transitive, R is an equivalence relation.

3. **Q3. Let f : R → {x ∈ R : −1 < x < 1} be defined by f(x) = x / (1 + |x|). Show that f is one-one and onto.**

Solution: First note the range: for any x ∈ R, |x/(1+|x|)| < 1, so f(x) always lies in (−1,1), confirming the codomain is correct. One-one: Consider two cases. Case A — both x_1, $x_2 \geq 0$: $f(x_1)=f(x_2) \Rightarrow x_1/(1+x_1) = x_2/(1+x_2) \Rightarrow x_1(1+x_2) = x_2(1+x_1) \Rightarrow x_1+x_1x_2 = x_2+x_1x_2 \Rightarrow x_1 = x_2$. Case B — both x_1, $x_2 < 0$: similarly f(x) = x/(1−x), and the same algebra gives $x_1 = x_2$. Case C — one positive, one negative (say $x_1>0>x_2$): then $f(x_1) > 0 > f(x_2)$, so $f(x_1) \neq f(x_2)$, no contradiction to rule out. Combining, $f(x_1)=f(x_2)$ forces $x_1=x_2$ in every case, so f is one-one. Onto: Let y ∈ (−1,1). If y ≥ 0, solve y = x/(1+x) (x≥0) ⇒ x = y/(1−y), which is ≥ 0 since 0≤y<1. If y < 0, solve y = x/(1−x) (x<0) ⇒ x = y/(1+y), which is < 0 since −1<y<0. In both cases a valid pre-image x ∈ R exists. Hence f is onto. Therefore f is one-one and onto (bijective).

4. **Q4. Let A = R − {3} and B = R − {1}. Consider f : A → B defined by f(x) = (x−2)/(x−3). Is f one-one and onto? Justify.**

Solution: One-one: Let $f(x_1) = f(x_2)$. Then $(x_1-2)/(x_1-3) = (x_2-2)/(x_2-3)$. Cross-multiplying: $(x_1-2)(x_2-3) = (x_2-2)(x_1-3)$. Expanding: $x_1x_2 -3x_1 -2x_2 +6 = x_1x_2 -3x_2 -2x_1 +6$. Cancel x_1x_2 and 6 from both sides: $-3x_1 -2x_2 = -3x_2 -2x_1 \Rightarrow -3x_1 + 2x_1 = -3x_2 + 2x_2 \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$. So f is one-one. Onto: Let y ∈ B (y ≠ 1). Set y = (x−2)/(x−3) and solve for x: y(x−3) = x−2 ⇒ yx − 3y = x − 2 ⇒ yx − x = 3y − 2 ⇒ x(y−1) = 3y−2 ⇒ x = (3y−2)/(y−1). Since y ≠ 1, this x is well defined. Also need x ≠ 3: if x = 3, then (3y−2)/(y−1) = 3 ⇒ 3y−2 = 3y−3 ⇒ −2 = −3, impossible, so x ≠ 3 always, i.e. x ∈ A. Thus every y ∈ B has a pre-image in A, so f is onto. Hence f is both one-one and onto, i.e. bijective.

5. **Q5. Show that the number of relations containing (1,2) and (2,3) which are reflexive and transitive but not symmetric, on A = {1,2,3}, is exactly three.**

Solution: For reflexivity we must include (1,1), (2,2), (3,3). We are given (1,2) and (2,3) must be included. For transitivity, since (1,2) and (2,3) are present, (1,3) must also be included (else transitivity fails). So the smallest possible relation is R_1 = {(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)}. Check R_1 is not symmetric: (1,2) ∈ R_1 but (2,1) ∉ R_1 — good, R_1 qualifies. Now, can we enlarge R_1 while staying reflexive, transitive, and non-symmetric? Adding (2,1) alone to R_1 gives $R_2 = R_1 \cup$ {(2,1)}; check transitivity: (2,1),(1,2) ∈ R_2 ⇒ need (2,2) ✓ present; (1,2),(2,1) ⇒ need (1,1) ✓ present — R_2 stays transitive, and it is still not symmetric because, e.g., (2,3) ∈ R_2 but (3,2) ∉ R_2. So R_2 works. Similarly adding only (3,2) to R_1 gives R_3, which can be checked to remain reflexive and transitive but not symmetric. However, adding both (2,1) and (3,2) together (or adding (3,1) alone in certain combinations) forces further pairs by transitivity/symmetry and eventually produces a symmetric (in fact universal) relation, which is excluded. Careful checking shows only R_1, R_2, R_3 satisfy all conditions. Hence the total number of such relations is three.

5. Homework Questions

Attempt these independently. Full solutions are provided below each question for self-checking — cover the answer while solving.

1. **HW1. Determine whether R = {(x, y) : y is divisible by x} on A = {1,2,3,4,5,6} is reflexive, symmetric, transitive.**

Answer: Reflexive: yes, since x is always divisible by x, so $(x,x) \in R$ for all x. Symmetric: no — e.g. $(2,4) \in R$ since 4 is divisible by 2, but $(4,2) \notin R$ since 2 is not divisible by 4. Transitive: yes — if y is divisible by x and z is divisible by y, then z is divisible by x (divisibility chains). So R is reflexive and transitive but not symmetric.

2. **HW2. Show that $R = \{(a,b) : a \le b^3\}$ in R is not reflexive, not symmetric, and not transitive (find suitable counter-examples).**

Answer: Not reflexive: take a = 1/2; is $1/2 \le (1/2)^3 = 1/8$? No. So $(1/2, 1/2) \notin R$. Not symmetric: take a=1, b=2; $1 \le 2^3=8$ true, so $(1,2) \in R$, but is $2 \le 1^3=1$? No, so $(2,1) \notin R$. Not transitive: take a=10, b=2, c=1; $10 \le 2^3=8$? False, so this triple doesn't even satisfy the hypothesis — try a=3, b=2, c=1: $3 \le 2^3=8$ true (so $(3,2) \in R$); $2 \le 1^3=1$ false (so hypothesis fails) — instead use a = 10, b = 3, c = 2: $10 \le 27$ true; $3 \le 8$ true; but $10 \le 2^3=8$ is false, so $(a,c) \notin R$ even though $(a,b),(b,c) \in R$. Hence R fails transitivity.

3. **HW3. Check the injectivity and surjectivity of f : Z → Z given by $f(x) = x^3$.**

Answer: One-one: $f(x_1)=f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ (cube function is strictly increasing, so it's injective on Z). Onto: not onto — e.g. $2 \in Z$ (codomain) has no integer cube root, since no integer cubes to 2. Hence f is one-one but not onto.

4. **HW4. Let A = {–1, 0, 1, 2}, B = {–4, –2, 0, 2}. $f(x) = x^2 - x$ and $g(x) = 2|x - 1/2| - 1$ for $x \in A$. Are f and g equal functions?**

Answer: Compute both at each point of A: at x=−1: $f=(-1)^2-(-1)=1+1=2$; $g=2|-1-0.5|-1=2(1.5)-1=2$. Equal. At x=0: $f=0-0=0$; $g=2|0-0.5|-1=2(0.5)-1=0$. Equal. At x=1: $f=1-1=0$; $g=2|1-0.5|-1=2(0.5)-1=0$. Equal. At x=2: $f=4-2=2$; $g=2|2-0.5|-1=2(1.5)-1=2$. Equal. Since $f(a)=g(a)$ for every $a \in A$, f and g are equal functions.

5. **HW5. Find the number of all one-one functions from the set {1,2,3} to itself, and explain the general formula for a set of size n.**

Answer: A one-one function from a finite set to itself is a permutation. For {1,2,3}, the number of one-one functions equals 3! = 6. In general, for a set with n elements, the number of one-one functions from the set to itself is n! (choose an image for the 1st element in n ways, the 2nd in (n−1) ways, and so on).

6. **HW6. Given a non-empty set X, define R on P(X) (the power set) by A R B iff $A \subset B$. Is R an equivalence relation? Justify.**

Answer: No, R is not an equivalence relation. Reflexive: yes, since $A \subset A$ for every subset A. Transitive: yes, since $A \subset B$ and $B \subset C \Rightarrow A \subset C$. But Symmetric: no — if $A \subsetneq B$ (A is a proper subset of B), then $A \subset B$ holds but $B \subset A$ fails (unless A = B). So R is reflexive and transitive but not symmetric, hence not an equivalence relation.

7. **HW7. Let f : R → R, f(x) = sin x and g : R → R, g(x) = cos x, restricted to [0, π/2]. Show f and g are individually one-one but f + g is not.**

Answer: On [0, π/2], sin x is strictly increasing and cos x is strictly decreasing, so both are one-one there (distinct x values give distinct sin or cos values). However, $(f+g)(0) = \sin 0 + \cos 0 = 0 + 1 = 1$, and $(f+g)(\pi/2) = \sin(\pi/2) + \cos(\pi/2) = 1 + 0 = 1$. Since $0 \ne \pi/2$ but $(f+g)(0) = (f+g)(\pi/2)$, the sum function f+g is not one-one, even though f and g individually are.

Homework Practice Set (attempt without looking up answers first)

- Show that R = {(x,y): x and y live in the same locality} on the set of humans in a town is an equivalence relation.
- Prove that f: R→R defined by $f(x)=1+x^2$ is neither one-one nor onto.
- Find gof and fog for $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, and state their domains.
- If f: A→B and g: B→C are both onto, show gof: A→C is onto.
- Find the number of onto functions from {1,2,3,4} to {1,2,3}.

— End of Study Material —